

A possible extension of lower and upper probabilities to the case of fuzzy random variables

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Starting point - Dempster's concept of interval-valued probabilities (1967)

- Suppose that $(\Omega, \mathcal{A}, \mathcal{P})$ is a probability space and that $X : \Omega \rightarrow \mathbb{R}$ is a random variable that can not be observed precisely.
- Instead of X it is possible to observe the interval $\tilde{X}(\omega) = [\underline{X}(\omega), \overline{X}(\omega)]$, which contains the true value $X(\omega)$ for every $\omega \in \Omega$.
- The random variable induces a probability measure \mathcal{P}^X on the Borel family $\mathcal{B}(\mathbb{R})$.
- **What can be said about the distribution \mathcal{P}^X of X by looking at the random interval \tilde{X} ?**

Starting point - Dempster's concept of interval-valued probabilities (1967)

- Assuming that \tilde{X} is a random interval it is possible to calculate lower and upper bounds $\underline{\pi}(B)$ and $\overline{\pi}(B)$ for the probability $\mathcal{P}^X(B)$ for every Borel set $B \in \mathcal{B}(\mathbb{R})$:

$$\begin{aligned}\underline{\pi}(B) &:= \mathcal{P}(\{\omega \in \Omega : \tilde{X}(\omega) \subseteq B\}) \\ \overline{\pi}(B) &:= \mathcal{P}(\{\omega \in \Omega : \tilde{X}(\omega) \cap B \neq \emptyset\})\end{aligned}\tag{1}$$

- Obviously $\underline{\pi}(B) \leq \mathcal{P}^X(B) \leq \overline{\pi}(B)$ holds for every Borel set $B \in \mathcal{B}(\mathbb{R})$.
- According to A. Dempster $\underline{\pi}$ and $\overline{\pi}$ are called *lower and upper probabilities induced by \tilde{X}* .

Starting point - Dempster's concept of interval-valued probabilities (1967)

- The same construction can also be done in the multidimensional setting - the random variable is replaced by a random vector and the containing interval is replaced by a containing random convex compact non-empty set.
- Regarding *fuzzy random variables as natural generalization of random intervals* it seems evident that the above concept can be applied directly - simply calculate lower and upper probabilities for every $\alpha \in (0, 1]$ and aggregate them in a suitable way.

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 - Fuzzy random vectors
- 3 The discrete case - fuzzy relative frequencies
 - Random set approach
- 4 Extension of upper and lower probabilities
 - Fuzzy probability distributions
 - Estimation of \mathbb{P}^* by means of fuzzy relative frequencies
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Definition (Fuzzy vector)

A d -dimensional fuzzy vector ξ^* is a real-valued function on \mathbb{R}^d with the following properties:

- 1 $0 \leq \xi^*(x) \leq 1 \quad \forall x \in \mathbb{R}^d$
- 2 $\forall \alpha \in (0, 1]$ the so-called α -cut $[\xi^*]_\alpha$, defined by $[\xi^*]_\alpha := \{x \in \mathbb{R}^d : \xi^*(x) \geq \alpha\}$, is a non-empty compact convex set.

If $d = 1$ then ξ^* is called *fuzzy number*.

- For every d -dimensional fuzzy vector ξ^* the support $\text{supp}(\xi^*)$ is defined by

$$\text{supp}(\xi^*) := \text{cls}(\{x \in \mathbb{R}^d : \xi^*(x) > 0\}). \quad (2)$$

- $\mathcal{F}_c(\mathbb{R}^d)$... d -dimensional fuzzy vectors
- $\mathcal{F}_{c,c}(\mathbb{R}^d)$... d -dimensional fuzzy vectors with compact support
- $\xi^* \oplus \eta^*$ denotes the common sum of two d -dimensional fuzzy vectors (fuzzy numbers), i.e.

$$[\xi^* \oplus \eta^*]_\alpha = [\xi^*]_\alpha + [\eta^*]_\alpha = \{x + y : x \in [\xi^*]_\alpha, y \in [\eta^*]_\alpha\}$$

- The following two types of metrics on $\mathcal{F}_{c,c}(\mathbb{R}^d)$ will be used:

$$\delta_{H,\infty}^*(\xi^*, \eta^*) := \sup_{\alpha \in (0,1]} \delta_H([\xi^*]_\alpha, [\eta^*]_\alpha) \quad (3)$$

$$\delta_{H,p}^*(\xi^*, \eta^*) := \left(\int_{(0,1]} (\delta_H([\xi^*]_\alpha, [\eta^*]_\alpha))^p d\alpha \right)^{1/p} \quad (4)$$

- If $(\Omega, \mathcal{A}, \mathcal{P})$ is a probability space and $X^* : \Omega \rightarrow \mathcal{F}_c(\mathbb{R}^d)$ ($d \geq 1$) is a fuzzy-valued function on Ω , then the following notation will be used for every $\omega \in \Omega$ and every $\alpha \in (0, 1]$:

$$X_\alpha(\omega) := [X^*(\omega)]_\alpha = \{x \in \mathbb{R}^d : (X^*(\omega))(x) \geq \alpha\} \quad (5)$$

Definition (Fuzzy random vector)

Let $(\Omega, \mathcal{A}, \mathcal{P})$ be a probability space, then a function $X^* : \Omega \rightarrow \mathcal{F}_c(\mathbb{R}^d)$ is called (*d-dimensional*) *fuzzy random vector* if

$$\{\omega \in \Omega : X_\alpha(\omega) \cap B \neq \emptyset\} \in \mathcal{A} \quad (6)$$

holds for every $B \in \mathcal{B}(\mathbb{R}^d)$ and every $\alpha \in (0, 1]$. If $d = 1$, then X^* is called *fuzzy random variable*.

(1) Fuzzy relative frequencies - Random set approach

- Given $A \subseteq \mathbb{R}^d$ and a d -dimensional fuzzy sample $x_1^*, x_2^*, \dots, x_n^*$.

- For $\alpha \in (0, 1]$ define

$$\underline{h}_{n,\alpha}(A) := \frac{1}{n} \# \left\{ i \in \{1, \dots, n\} : [x_i^*]_\alpha \subseteq A \right\} \quad (\text{contained})$$

$$\bar{h}_{n,\alpha}(A) := \frac{1}{n} \# \left\{ i \in \{1, \dots, n\} : [x_i^*]_\alpha \cap A \neq \emptyset \right\} \quad (\text{hitting}).$$

- This yields a nested family of closed intervals

$$h_{n,\alpha}(A) := [\underline{h}_{n,\alpha}(A), \bar{h}_{n,\alpha}(A)] \quad \text{for every } \alpha \in (0, 1],$$

that in turn induces a unique fuzzy number (the *convex hull* of the family $h_{n,\alpha}(A)$), which will be denoted by $h_n^*(A)$.

Figure: Fuzzy sample of size 10

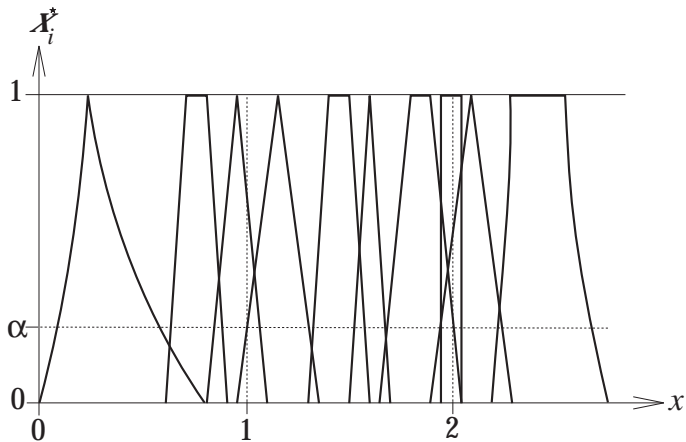
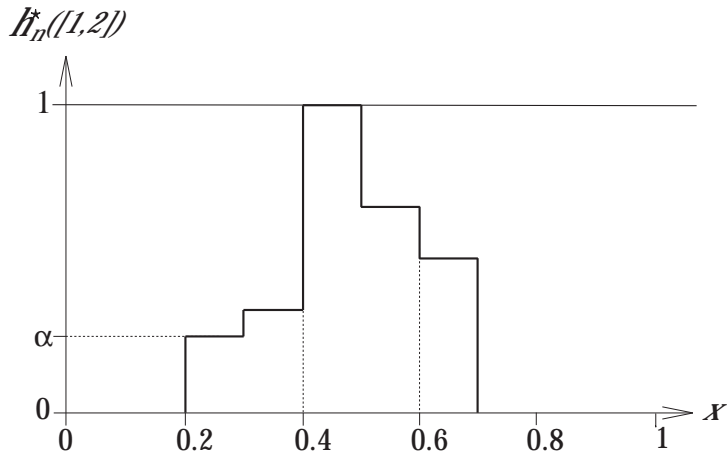


Figure: Fuzzy relative frequency $h_n^*([1, 2])$ 

Lemma (Basic properties of $h_n^*(\cdot)$ as set-mapping)

- $\underline{h}_{n,\alpha}(\mathbb{R}^d) = \bar{h}_{n,\alpha}(\mathbb{R}^d) = 1 \quad \forall \alpha \in (0, 1]$
- $\underline{h}_{n,\alpha}(\emptyset) = \bar{h}_{n,\alpha}(\emptyset) = 0, \quad \forall \alpha \in (0, 1]$
- $\underline{h}_{n,\alpha}(A), \bar{h}_{n,\alpha}(A) \in [0, 1] \quad \forall A \subseteq \mathbb{R}^d, \alpha \in (0, 1]$
- $\underline{h}_{n,\alpha}(\cdot)$ and $\bar{h}_{n,\alpha}(\cdot)$ are monotonic set functions for every $\alpha \in (0, 1]$
- $\underline{h}_{n,\alpha}(\cdot)$ is a superadditive and $\bar{h}_{n,\alpha}(\cdot)$ is a subadditive set function for every $\alpha \in (0, 1]$
- ...

Fuzzy probability distributions

- Suppose that $(\Omega, \mathcal{A}, \mathcal{P})$ is a probability space and that $X^* : \Omega \rightarrow \mathcal{F}_c(\mathbb{R}^d)$ is a d -dimensional fuzzy random vector.
- X^* induces families $(\underline{\pi}_\alpha)_{\alpha \in (0,1]}$ and $(\overline{\pi}_\alpha)_{\alpha \in (0,1]}$ of upper and lower probabilities on $\mathcal{B}(\mathbb{R}^d)$ (in Dempster's sense) by

$$\begin{aligned}\underline{\pi}_\alpha(A) &:= \mathcal{P}(\{\omega \in \Omega : X_\alpha(\omega) \subseteq A\}) \\ \overline{\pi}_\alpha(A) &:= \mathcal{P}(\{\omega \in \Omega : X_\alpha(\omega) \cap A \neq \emptyset\}).\end{aligned}\tag{7}$$

- These families can be aggregated to a unique fuzzy number $\mathbb{P}^*(A) \in \mathcal{F}_{c,c}(\mathbb{R})$ (again the convex hull) called *fuzzy probability of the set A*.

The $\mathcal{F}_{c,c}(\mathbb{R})$ -valued set function $\mathbb{P}^*(\cdot)$, defined on $\mathcal{B}(\mathbb{R}^d)$, is called *fuzzy probability distribution induced by X^** .

Lemma

Suppose that $(\Omega, \mathcal{A}, \mathcal{P})$ is a probability space and that $X^* : \Omega \rightarrow \mathcal{F}_c(\mathbb{R}^d)$ is a d -dimensional fuzzy random vector. Then the upper and lower probabilities $\underline{\pi}_\alpha$ and $\bar{\pi}_\alpha$ fulfill the following properties for every $\alpha \in (0, 1]$

- $\underline{\pi}_\alpha(\mathbb{R}^d) = \bar{\pi}_\alpha(\mathbb{R}^d) = 1$ $\underline{\pi}_\alpha(\emptyset) = \bar{\pi}_\alpha(\emptyset) = 0$
- $\underline{\pi}_\alpha(A^c) = 1 - \bar{\pi}_\alpha(A)$ holds for every $A \in \mathcal{B}(\mathbb{R}^d)$
- $\underline{\pi}_\alpha(\cdot)$ is a superadditive and $\bar{\pi}_\alpha(\cdot)$ is a subadditive set function
- $\underline{\pi}_\alpha(\cdot)$ and $\bar{\pi}_\alpha(\cdot)$ are monotonic set functions
- $\bar{\pi}_\alpha(\cdot)$ is a completely alternating, \mathcal{C} -convex Choquet capacity on $\mathcal{K}_c(\mathbb{R}^d)$

- The precise definition of $\mathbb{P}^*(A) \in \mathcal{F}_{c,c}(\mathbb{R})$ is

$$(\mathbb{P}^*(A))(x) := \sup \{ \alpha \in (0, 1] : x \in ([\underline{\pi}_\alpha(A), \bar{\pi}_\alpha(A)]) \}. \quad (8)$$

- This is easily seen to be equivalent to setting

$$[\mathbb{P}^*(A)]_\alpha = [\underline{p}_\alpha(A), \bar{p}_\alpha(A)] = \bigcap_{\beta < \alpha} [\underline{\pi}_\beta(A), \bar{\pi}_\beta(A)]. \quad (9)$$

- Consequently all the properties stated in Lemma 4 also hold for the α -cuts of \mathbb{P}^* .

Motivation

- **Well known:** Given a sequence of pairwise independent identically distributed integrable random variables $(X_n)_{n \in \mathbb{N}}$ on $(\Omega, \mathcal{A}, \mathcal{P})$ the classical relative frequency $h_n(A, \cdot)$ converges with probability 1 for $n \rightarrow \infty$ to $\mathcal{P}(X \in A)$.
- **Question:** Does a similar result also hold for fuzzy relative frequencies?
 - I.e. if $(X_n^*)_{n \in \mathbb{N}}$ is a sequence of pairwise independent identically distributed fuzzy random vectors on $(\Omega, \mathcal{A}, \mathcal{P})$ does the fuzzy relative frequency $h_n^*(A, \cdot)$ converge for $n \rightarrow \infty$?
 - If yes, in which sense (with respect to which metric?)

- Given a sequence $(X_n^*)_{n \in \mathbb{N}}$ of d -dimensional fuzzy random vectors on a probability space $(\Omega, \mathcal{A}, \mathcal{P})$ define the fuzzy relative frequency $h_n^*(A, \omega)$ of the Borel set $A \in \mathcal{B}(\mathbb{R}^d)$ with respect to the sample $(X_n^*(\omega))_{n \in \mathbb{N}}$ analogous to before as the convex hull of the family $[\underline{h}_{n,\alpha}(A, \omega), \bar{h}_{n,\alpha}(A, \omega)]$, with

$$\underline{h}_{n,\alpha}(A, \omega) := \frac{1}{n} \# \left\{ i \in \{1, \dots, n\} : [X_i^*(\omega)]_\alpha \subseteq A \right\}$$

$$\bar{h}_{n,\alpha}(A, \omega) := \frac{1}{n} \# \left\{ i \in \{1, \dots, n\} : [X_i^*(\omega)]_\alpha \cap A \neq \emptyset \right\}.$$

- X^* and Y^* are called *independent*, iff the following equality holds for every $\alpha \in (0, 1]$ and all Borel sets $B_1, B_2 \in \mathcal{B}(\mathbb{R}^d)$:

$$\mathcal{P}(X_\alpha \subseteq B_1, Y_\alpha \subseteq B_2) = \mathcal{P}(X_\alpha \subseteq B_1) \mathcal{P}(Y_\alpha \subseteq B_2)$$

Theorem (Convergence of fuzzy relative frequencies)

Suppose that X^, X_1^*, X_2^*, \dots is a sequence of pairwise independent, identically distributed d -dimensional fuzzy random vectors and that $A \in \mathcal{B}(\mathbb{R}^d)$ is an arbitrary Borel set.*

Then with probability 1 the fuzzy relative frequency $h_n^(A, \cdot)$ converges to $\mathbb{P}^*(A)$ with respect to the metric $\delta_{H, \infty}^*$ and with respect to the metric $\delta_{H, p}^*$ for every $p \geq 1$.*

Corollary (Consistency)

Under the above assumptions the fuzzy relative frequency $h_n^(A, \cdot)$ is a strongly consistent estimator for the fuzzy-valued probability $\mathbb{P}^*(A)$.*

Open points

- Is it possible to prove a Glivenko-Cantelli-like Theorem in the following way (same setting as before with $d = 1$):

$$\sup_{x \in \mathbb{R}} D\left(h_n^*((-\infty, x]), \mathbb{P}^*((-\infty, x])\right) \rightarrow 0 \quad \text{a.s.}$$

- The presented concept of fuzzy relative frequencies has some disadvantages - in general a sample can not be reconstructed by looking at the fuzzy empirical distribution function $F_n^*(x) := h_n^*((-\infty, x])$.

Motivation

Notation and preliminaries

The discrete case - fuzzy relative frequencies

Extension of upper and lower probabilities

Open points

Thank you for your attention !